GSQ/D-20 **1054**

MATHEMATICS
BM-352
Groups and Rings

Time : Three Hours] [Maximum Marks : 40

Note : Attempt *Five* questions in all, selecting *one* question from each Section. Q. No. **1** is compulsory.

(Compulsory Question)

1. (a) Prove that every subgroup of an abelian group is always normal. **1½**

(b) Prove that identity mapping is the only inner automorphism for an abelian group. **1½**

(c) Let $f : R \to R'$ be a homomorphism. Then f is one to one if kerf = $\{0\}$. **1½**

(d) Define Euclidean ring. **1½**

(e) Define transposition. What do you mean by even and odd permutations ? **2**

Section I

2. (a) Prove that order of every element of a finite group is finite and is less than or equal to the order of the group. **4**

(b) Prove that every subgroup of a cyclic group is cyclic. **4**

3. (a) Prove that the order of every element of a finite group is a divisor of the order of the group. **4**

(b) If a group (G, $\cdot$) has four elements, show that it must be abelian. **4**

Section II

4. (a) Prove that the set Inn(G) of all inner automorphisms of a group G is isomorphic to the quotient group G/Z(G), where Z(G) is the centre of G. **4**

(b) Let $f : G \to G$ be a homomorphism. Let f commutes with every inner automorphism of G. Show that $H = \{x \in G; f^2(x) = f(x)\}$ is a normal subgroup of G. **4**

5. (a) Let G' be commutator subgroup of a group G. Then G is abelian iff $G' = \{e\}$, where e is the identity element of G. **4**

(b) Find the centre of the permutation group S_3. **4**

Section III

6. (a) Show that every field is an integral domain. Also show by an example that every integral domain need not be a field. **4**

(b) Let R be a commutative ring. An ideal S of R is a prime ideal iff for two ideals A, B of R, AB $\subseteq$ S $\Rightarrow$ either A $\subseteq$ S or B $\subseteq$ S. **4**

7. (a) Show that an ideal S of a commutative ring R with unity is maximal iff R/S is a field. **4**

(b) Let f be a ring isomorphism of R onto R′. show that if R′ is an integral domain, then so is R. **4**

Section IV

8. (a) Show that an element in a principal ideal domain is prime element iff it is irreducible. **4**

(b) Show that $\sqrt{-5}$ is a prime element of the ring $z\sqrt{-5} = \left\{a + b\sqrt{-5} : a, b \in \mathrm{Z}\right\}$. **4**

9. (a) Prove that every principal ideal domain is a unique factorization domain. **4**

(b) Show that the polynomial : **4**

$$1 + x + x^2 + x^3 + x^4$$

is irreducible over Q.