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GSM/J-21 1580

MATHEMATICS

BM-241

Sequences and Series

Time : Three Hours] [Maximum Marks : 27

Note : Attempt *Five* questions in all, selecting *one* question from each Section. Q. No. **1** is compulsory.

Compulsory Question

1. (a) Show that the set I of irrational numbers is not a neighbourhood of any real number. **1**

(b) Give an example of a finitely oscillating sequence. **1**

(c) Prove that Greatest lower bound of a set, if it exists, is unique. **1**

(d) State Cauchy's root test for a series. **1**

(e) Show that the series $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots$ converges to 1. **1**

(f) State Dirichlets test for the convergence of arbitrary series. **1**

(g) Show that the infinite product $\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)$ is divergent. **1**

Section I

2. (a) If S and T are non-empty bounded subsets of R, then prove that $S \cup T$ is also bounded and : **2½**

$$\sup(S \cup T) = \max\{\sup S, \sup T\}$$

(b) Prove that the intersection of a finite number of open sets is an open set. **2½**

3. (a) If A and B are subsets of R, then : **2½**

$$(A \cup B)' = A' \cup B'$$

(b) Prove that the derived set of any set is a closed set. **2½**

Section II

4. (a) If $\langle a_n \rangle$ is a sequence of positive terms and $\lim_{n\to\infty} \langle a_n \rangle^{1/n}$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ both exist finitely or infinitely, then prove that : **2½**

$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

(b) Prove that the sequence $\langle a_n \rangle$ defined by $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$ converges to the positive root of the equation $x^2 - x - 2 = 0$. **2½**

5. (a) Show that the sequence $\left\langle 1 + \frac{1}{2} + \frac{1}{3} + \dots\dots\dots + \frac{1}{n} \right\rangle$ is not convergent, while :

$$\left\langle \frac{1}{n}\left(1 + \frac{1}{2} + \frac{1}{3} + \dots\dots\dots + \frac{1}{n}\right) \right\rangle$$

is convergent. **2½**

(b) Discuss the convergence of the series : **2½**

$$\sum_{n=1}^{\infty} \sqrt{n^4 + 1} - \sqrt{n^4 - 1}$$

Section III

6. (a) Examine the convergence or divergence of the following series with $x > 0$: **2½**

$$\frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \dots.$$

(b) Test the convergence of the series : **2½**

$$\sum_{n=1}^{\infty} \frac{1.3.5.......(2n-1)}{2.4.6.........2n} \cdot \frac{x^{2n+1}}{2n+1}, (x > 0)$$

7. (a) Using Cauchy's condensation test, discuss the convergence of the series : **2½**

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$$

(b) State and prove Cauchy's Integral test for the convergence of infinite series. **2½**

Section IV

8. (a) Test the convergence and absolute convergence of the series : **2½**

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n+1}}$$

(b) If $\sum_{n=1}^{\infty} a_n$ is convergent and the sequence $< b_n >$ is monotonic and bounded, then prove that $\sum_{n=1}^{\infty} a_n b_n$ is convergent. **2½**

9. (a) Show that the Cauchy product of the convergent series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ with itself is not convergent. **2½**

(b) Prove that the infinite product $\prod_{n=1}^{\infty} \left(1+\frac{x}{n}\right) e^{-x/n}$ is absolutely convergent for all real x. **2½**